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Extended (2, 4)-Designs

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In this paper, the concept of an extended (2, 4)-design is introduced. An extended (2, 4)-design is a pair $(X, \mathscr{B})$ where X is a finite set and $\mathscr{B}$ is a collection of 4-tuples of not necessarily distinct elements of X, such that every pair of not necessarily distinct elements of X is contained in exactly one member of $\mathscr{B}$. It is shown that an extended (2,4)-design of order n exists for every positive integer n except $n = 6$, 8 and 9. Several inequivalent designs of order n are obtained.

1. INTRODUCTION

A (2, 4)-*design* is a pair (X, B) where X is a finite set and B is a collection of 4-subsets of X, called 4-*tuples*, such that every pair of distinct elements of X is contained in exactly one 4-tuple of B. The number $|X|$ is called the order of the (2, 4)-design (X, B). A (2, 4)-design of order v is a balanced incomplete block design (BIBD) with parameters v, $k = 4$ and $\lambda = 1$. Hanani (see [5]) has shown that these designs exist if and only if $v \equiv 1$ or 4 (mod 12). A (2, 4)-*covering* (or, respectively, (2, 4)-*packing*) is a pair (X, F) where X is a finite set and F is a collection of 4-tuples of X such that every pair of distinct elements of X is contained in at least (at most) one 4-tuple of F. A (2, 4)-covering (X, F) is called *minimal* if $|F|$ is minimal. These designs have been constructed by Mills [8, 9]. A (2, 4)-packing (X, F) is called *maximal* if $|F|$ is maximal. Maximal (2, 4)-packings have been constructed by Brouwer [2].

In this paper we introduce the concept of an *extended* (2, 4)-*design*. An extended (2, 4)-design is a pair $(X, \mathscr{B})$ where X is a finite set and $\mathscr{B}$ is a collection of 4-tuples of *not necessarily distinct* elements of X, such that every pair of *not necessarily distinct* elements of X is contained in exactly one

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member of $\mathscr{B}$. This definition is analogous to that given for extended triple systems (cf. [1, 7]). We shall show that an extended (2, 4)-design of order n exists for every positive integer n except $n = 6$, 8 and 9. Several inequivalent designs are constructed for a given value of n.

2. Preliminaries and Auxiliary Designs

The 4-tuples of an extended (2, 4)-design $(X, \mathscr{B})$ are essentially of five types: (1) (a, b, c, d), (2) (a, a, b, c), (3) (a, a, b, b), (4) (a, a, a, b) and (5) (a, a, a, a), where the elements a, b, c, d are pairwise distinct. In what follows we shall let α_0, α_2, $\alpha_{2,2}$, α_3, α_4 denote, respectively, the *number* of 4-tuples of $(X, \mathscr{B})$ that are of type (1), (2), (3), (4), (5). A simple counting argument shows that if $(X, \mathscr{B})$ is of order n, then

$$\alpha_0 = \frac{1}{6}\left(\binom{n}{2} - 3\alpha_2 - \alpha_{2,2} - \alpha_3\right), \tag{2.1}$$

$$\alpha_4 = n - \alpha_2 - 2\alpha_{2,2} - \alpha_3 . \tag{2.2}$$

Evidently, α_0 and α_4 are determined by α_2, $\alpha_{2,2}$ and α_3. We shall let $\{n; \alpha_2, \alpha_{2,2}, \alpha_3\}$ denote the class of all extended (2, 4)-designs of order n with the parameters α_2, $\alpha_{2,2}$, and α_3. We say that $\{n; \alpha_2, \alpha_{2,2}, \alpha_3\}$ exists if there is a design with the specified parameters. Concerning the existence of $\{n; \alpha_2, \alpha_{2,2}, \alpha_3\}$, we have

Lemma 2.1. *A necessary condition for the existence of* $\{n; \alpha_2, \alpha_{2,2}, \alpha_3\}$ *is that* (i) $\binom{n}{2} - 3\alpha_2 - \alpha_{2,2} - \alpha_3 \equiv 0 \pmod 6$ *and* (ii) $0 \leqslant \alpha_2 + \alpha_{2,2} + \alpha_3 \leqslant n$.

Proof. The proof of (i) follows from (2.1) and (ii) follows from (2.2) and the fact that $\alpha_4 \geqslant 0$.

Before proceeding to our construction of extended (2, 4)-designs, we shall introduce some auxiliary designs. The following result is contained in [9].

Lemma 2.2. *If* $n \equiv 7$ *or* $10 \pmod{12}$, $n \neq 7, 10, 19$, *then there is a minimal* (2, 4)-*covering* (X, F) *of order* n *such that one pair occurs four times in the 4-tuples of* F, *while all other pairs occur exactly once.*

Let K and M be sets of positive integers. A *pairwise balanced design* (PBD) $B(K, 1; v)$ is a pair (X, B) where X is a v-set (of *points*) and B is a collection of subsets of X (called *blocks*) with sizes in K such that every block contains at least two points and every pair of distinct points of X is contained in exactly one block of B. A *group divisible design* (GDD) $GD(K, 1, M; v)$ is a pairwise balanced design $B(K \cup M, 1; v)$ with a distinguished parallel class of blocks (called *groups*) with sizes in M, while all other blocks (called blocks of the group divisible design) have sizes in K. We shall write $B(k, 1; v)$ for $B(\{k\}, 1; v)$ and similarly $GD(k, 1, m; v)$ for $GD(\{k\}, 1, \{m\}; v)$. An asterisk appearing on an element of K or M indicates there is exactly one block or group of that size in the PBD or GDD. We observe that a PBD $B(k, 1; v)$ is a BIBD with parameters $v, k, \lambda = 1$. A more general concept of PBD's and GDD's can be found in [5]. The following three lemmas are contained in [2, 3].

LEMMA 2.3. *A pairwise balanced design $B(\{4, 7^*\}, 1; v)$ exists if and only if $v \equiv 7$ or $10 \pmod{12}$, $v \neq 10, 19$.*

LEMMA 2.4. *A group divisible design $GD(4, 1, 2; v)$ exists if and only if $v \equiv 2$ or $8 \pmod{12}$, $v \neq 8$.*

LEMMA 2.5. *A group divisible design $GD(4, 1, \{2, 5^*\}; v)$ exists if and only if $v \equiv 5$ or $11 \pmod{12}$, $v \neq 11, 17$.*

In some of our constructions we shall use Hall's Theorem [4, p. 45] on systems of distinct representatives in the form

LEMMA 2.6. *Let I be a finite set of indices, $I = \{1, 2,..., n\}$. For each $i \in I$ let S_i be a subset of a set S. A necessary and sufficient condition for the existence of distinct representatives x_i, $i = 1, 2,..., n$, $x_i \in S_i$, $x_i \neq x_j$ when $i \neq j$ is that for every $k = 1, 2,..., n$ and choice of k distinct indices $i_1, i_2,..., i_k$, the subsets $S_{i_1}, S_{i_2},..., S_{i_k}$ contain between them at least k distinct elements.*

The following basic lemma will prove quite useful in deriving extended (2, 4)-designs from PBD's.

LEMMA 2.7. *Let (X, B) be a pairwise balanced design $B(\{2, 3, 4\}, 1; n)$. Let B_1 and B_2 denote, respectively, the collection of blocks of B of sizes two and three. Let $X^* \subseteq X$ be the of all the points covered by the blocks of B_1 and B_2. If the blocks of B_1 and B_2, considered as subsets of X^*, possess distinct representatives, then $\{n; n_1, 0, n_2\}$ exists, where $n_1 = |B_2|$ and $n_2 = |B_1|$.*

Proof. Under the hypothesis of the lemma, we can derive from the PBD (X, B) an extended (2, 4)-design $(X, \mathscr{B})$ where the collection $\mathscr{B}$ consists of (i) the blocks in $B - (B_1 \cup B_2)$, (ii) the blocks (a, a, a, b) where a is the representative of the block $(a, b) \in B_1$, (iii) the blocks (x, x, y, z) where x is the representative of the block $(x, y, z) \in B_2$, (iv) the blocks (i, i, i, i) where $i \in X$ is not a representative of a block in B_1 or B_2. It is readily checked that $(X, \mathscr{B}) \in \{n; n_1, 0, n_2\}$.

Let $(X, \mathscr{B})$ and $X^*, \mathscr{B}^*)$ be two extended (2, 4)-designs. If $X \subseteq X^*$ and $\mathscr{B} \subseteq \mathscr{B}^*$, we shall say that $(X, \mathscr{B})$ is a *subdesign* of $(X^*, \mathscr{B}^*)$ and that $(X^*, \mathscr{B}^*)$ contains $(X, \mathscr{B})$. If $\mathscr{B} \cap \mathscr{B}^* = \varnothing$, we say that $(X, \mathscr{B})$ and $(X^*, \mathscr{B}^*)$ are *disjoint*. If there is a bijection $\varphi: X \to X^*$ such that $(\mathscr{B})\varphi = \mathscr{B}^*$, then we say that $(X, \mathscr{B})$ and $(X^*, \mathscr{B}^*)$ are *isomorphic* (or *equivalent*).

The following lemma is concerned with the removal and replacement of subdesigns and will be quite useful.

LEMMA 2.8. *Suppose there is an extended* (2, 4)-*design* $(X, \mathscr{B}) \in \{n; \alpha_2, \alpha_{2,2}, \alpha_3\}$ *containing a subdesign* $(X', \mathscr{B}') \in \{m; \alpha_2', \alpha_{2,2}', \alpha_3'\}$. *If there is an extended* (2, 4)-*design* $(X', \mathscr{B}'') \in \{m; \alpha_2'', \alpha_{2,2}'', \alpha_3''\}$, *then there exists an extended* (2, 4)-*design* $(X, \mathscr{B}^*)$ *in* $\{n; \alpha_2 - \alpha_2' + \alpha_2'', \alpha_{2,2} - \alpha_{2,2}' + \alpha_{2,2}'', \alpha_3 - \alpha_3' + \alpha_3''\}$ *containing* $(X', \mathscr{B}'')$.

Proof. The subdesign $(X', \mathscr{B}')$ of $(X, \mathscr{B})$ can be removed and replaced by $(X', \mathscr{B}'')$ and the result follows immediately.

A BIBD $B(k, 1; v)$ is called *resolvable* if its blocks can be partitioned into parallel classes. A resolvable $B(3, 1; v)$ is called a Kirkman triple system and such a design exists if and only if $v \equiv 3 \pmod 6$ (see [10]). It is also known [6] that a resolvable $B(4, 1; v)$ exists if and only if $v \equiv 4 \pmod{12}$. From the existence of a resolvable $B(3, 1; v)$ we may obtain the following lemma:

LEMMA 2.9. *Let t be a positive integer and suppose there exists an extended* (2, 4)-*design* $(X, \mathscr{B})$ *in* $\{3t + 1; \alpha_2, \alpha_{2,2}, \alpha_3\}$, *then there exists an extended* (2, 4)-*design* $(X^*, \mathscr{B}^*)$ *in* $\{9t + 4; \alpha_2, \alpha_{2,2}, \alpha_3\}$ *containing* $(X, \mathscr{B})$.

Proof. Let the design $(X, \mathscr{B})$ be based on the set $X = \{\infty_i \mid 1 \leqslant i \leqslant 3t + 1\}$. Let $X^* = X \cup Y$, where $Y = \{1, 2, \ldots, 6t + 3\}$ and $X \cap Y = \varnothing$. Let (Y, B) be a resolvable $B(3, 1; 6t + 3)$ and let $B = \bigcup_{i=1}^{3t+1} B_i$ be a partition into parallel classes. Let $\mathscr{B}' = \{(x, x, x, x) \mid x \in Y\}$ and for $1 \leqslant i \leqslant 3t + 1$ define $\mathscr{B}_i = \{(a, b, c, \infty_i) \mid (a, b, c) \in B_i\}$. Put $\mathscr{B}^* = \mathscr{B} \cup \mathscr{B}' \cup \mathscr{B}_1 \cup \mathscr{B}_2 \cup \cdots \cup \mathscr{B}_{3t+1}$. It is readily checked that $(X^*, \mathscr{B}^*)$ satisfies the desired properties.

It is perhaps worth noting that if $t > 1$ in Lemma 2.9, then the subdesign $(X, \mathscr{B})$ of $(X^*, \mathscr{B}^*)$ is unique in the sense that it is the only subdesign of that order contained in $(X^*, \mathscr{B}^*)$.

3. EXTENDED (2, 4)-DESIGNS OF SMALL ORDER

Up to isomorphism there is only one extended (2 4)-design of order n for $n = 1, 7$. There are two non-isomorphic extended (2, 4)-designs of order n for $n = 2, 3, 4, 5$. It will be shown in this section that there are no extended

(2, 4)-designs of order n for $n = 6, 8, 9$. First of all, we label the designs of order $n = 1, 2, 3, 4, 5, 7$ for future reference and convenience.

$D_1 = \{(1, 1, 1, 1)\} \in \{1; 0, 0, 0\}$.

$D_2 = \{(1, 1, 1, 1), (2, 2, 2, 1)\} \in \{2; 0, 0, 1\}$.

$D_2^* = \{(1, 1, 2, 2)\} \in \{2; 0, 1, 0\}$.

$D_3 = \{(1, 1, 2, 3), (2, 2, 2, 2), (3, 3, 3, 3)\} \in \{3; 1, 0, 0\}$.

$D_3^* = \{(1, 1, 1, 2), (2, 2, 2, 3), (3, 3, 3, 1)\} \in \{3; 0, 0, 3\}$.

$D_4 = \{(1, 1, 1, 1), (2, 2, 2, 2), (3, 3, 3, 3), (4, 4, 4, 4), (1, 2, 3, 4)\} \in \{4; 0, 0, 0\}$.

$D_4^* = \{(1, 1, 2, 3), (2, 2, 2, 4), (3, 3, 3, 4), (4, 4, 4, 1)\} \in \{4; 1, 0, 3\}$.

$D_5 = \{(1, 1, 1, 1), (2, 2, 2, 1), (3, 3, 3, 1), (4, 4, 4, 1), (5, 5, 5, 1), (2, 3, 4, 5)\} \in \{5; 0, 0, 4\}$.

$D_5^* = \{(1, 1, 2, 2), (3, 3, 3, 1), (4, 4, 4, 1), (5, 5, 5, 1), (2, 3, 4, 5)\} \in \{5; 0, 1, 3\}$.

$D_7 = \{(1, 1, 2, 4), (2, 2, 3, 5), (3, 3, 4, 6), (4, 4, 5, 7), (5, 5, 6, 1), (6, 6, 7, 2), (7, 7, 1, 3)\} \in \{7; 7, 0, 0\}$.

Concerning the non-existence of designs in $\{n; \alpha_2, \alpha_{2,2}, \alpha_3\}$ for $n = 6, 8, 9$, we shall observe an inequality arising from (2.1). Let $D(2, 4, n)$ denote the number of 4-tuples in a maximal (2, 4)-packing of order n. $D(2, 4, n)$ is given explicitly for all values of n in [2]. Let $\lceil x \rceil$ denote the smallest integer that is at least x. Then we have

LEMMA 3.1. *A necessary condition for the existence of* $\{n; \alpha_2, \alpha_{2,2}, \alpha_3\}$ *is that* $\lceil n(n-7)/12 \rceil \leqslant \alpha_0 \leqslant D(2, 4, n)$.

Proof. By the definition of a maximal (2, 4)-packing of order n it is clear that $\alpha_0 \leqslant D(2, 4, n)$. For the other inequality, we use the fact that $0 \leqslant \alpha_2 + \alpha_{2,2} + \alpha_3 \leqslant n$. Consequently, we obtain $\alpha_0 \geqslant \frac{1}{6}(\binom{n}{2} - 3n)$ and since α_0 is a non-negative integer, $\alpha_0 \geqslant \lceil n(n-7)/12 \rceil$ as required.

In what follows we note that $D(2, 4, 6) = 1$, $D(2, 4, 8) = 2$, $D(2, 4, 9) = 3$.

THEOREM 3.2. $\{6; \alpha_2, \alpha_{2,2}, \alpha_3\}$ *does not exist for any admissible values of* $\alpha_2, \alpha_{2,2}, \alpha_3$.

Proof. Suppose there is a design $(X, \mathscr{B})$ in $\{6; \alpha_2, \alpha_{2,2}, \alpha_3\}$. Let $X = \{1, 2,..., 6\}$. By (2.1) and Lemmas 2.1 and 3.1, we have

(i) $3\alpha_2 + \alpha_{2,2} + \alpha_3 = 15 - 6\alpha_0$,

(ii) $0 \leqslant \alpha_2 + \alpha_{2,2} + \alpha_3 \leqslant 6$,

(iii) $0 \leqslant \alpha_0 \leqslant 1$.

In addition, it is easy to determine that $\alpha_2 \leqslant 4$. From (iii) we have $\alpha_0 = 0$ or 1. If $\alpha_0 = 0$, then by (i) $\alpha_2 \leqslant 4$ implies that $\alpha_2 + \alpha_{2,2} + \alpha_3 = 15 - 2\alpha_2 \geqslant 15 - 8 = 7$, which is impossible by (ii). If $\alpha_0 = 1$, then we may assume without loss of generality that $\mathscr{B}$ contains the 4-tuple (1, 2, 3, 4). However, the pair (5, 6) must appear in exactly one 4-tuple of $\mathscr{B}$ and it is evident that $\alpha_2 = 1$. From (i) $\alpha_2 + \alpha_{2,2} + \alpha_3 = 9 - 2\alpha_2 \geqslant 9 - 2 = 7$, which is impossible again. This completes the proof.

THEOREM 3.3. *$\{8; \alpha_2, \alpha_{2,2}, \alpha_3\}$ does not exist for any admissible values of $\alpha_2, \alpha_{2,2}, \alpha_3$.*

Proof. Suppose there is a design $(X, \mathscr{B})$ in $\{8; \alpha_2, \alpha_{2,2}, \alpha_3\}$. Let $X = \{1, 2,..., 8\}$. Then by (2.1) and Lemmas 2.1 and 3.1, we have

(i) $3\alpha_2 + \alpha_{2,2} + \alpha_3 = 28 - 6\alpha_0$,

(ii) $0 \leqslant \alpha_2 + \alpha_{2,2} + \alpha_3 \leqslant 8$,

(iii) $1 \leqslant \alpha_0 \leqslant 2$.

From (iii) $\alpha_0 = 1$ or 2.

If $\alpha_0 = 1$, we may assume that $\mathscr{B}$ contains the 4-tuple (1, 2, 3, 4). In this case, each of the α_2 4-tuples of $\mathscr{B}$ must contain a pair from the set {5, 6, 7, 8} and so $\alpha_2 \leqslant 6$. From (i) it follows that $\alpha_2 + \alpha_{2,2} + \alpha_3 = 22 - 2\alpha_2 \geqslant 22 - 12 = 10$, which is impossible by (ii). If $\alpha_0 = 2$ and $\mathscr{B}$ contains two disjoint 4-tuples (1, 2, 3, 4) and (5, 6, 7, 8), then $\alpha_2 = 0$. From (i) we would obtain $\alpha_2 + \alpha_{2,2} + \alpha_3 = 28 - 12 = 16$, which is impossible by (ii). If $\alpha_0 = 2$ and $\mathscr{B}$ contains the 4-tuples (1, 2, 3, 4) and (1, 5, 6, 7), then it is easily determined that $\alpha_2 \leqslant 3$. Thus from (i) we have $\alpha_2 + \alpha_{2,2} + \alpha_3 = 16 - 2\alpha_2 \geqslant 16 - 6 = 10$, which is again impossible. This completes the proof of the theorem.

THEOREM 3.4. *$\{9; \alpha_2, \alpha_{2,2}, \alpha_3\}$ does not exist for any admissible values of $\alpha_2, \alpha_{2,2}, \alpha_3$.*

Proof. Suppose there is a design $(X, \mathscr{B})$ in $\{9; \alpha_2, \alpha_{2,2}, \alpha_3\}$. Let $X = \{1, 2,..., 9\}$. From (2.1) and Lemmas 2.1 and 3.1, we have

(i) $3\alpha_2 + \alpha_{2,2} + \alpha_3 = 36 - 6\alpha_0$,

(ii) $0 \leqslant \alpha_2 + \alpha_{2,2} + \alpha_3 \leqslant 9$,

(iii) $2 \leqslant \alpha_0 \leqslant 3$.

It follows that $\alpha_0 = 2$ or 3. If $\alpha_0 = 2$ and we assume that $\mathscr{B}$ contains two disjoint 4-tuples (1, 2, 3, 4) and (5, 6, 7, 8), then it is easily checked that $\alpha_2 \leqslant 4$. In this case we obtain from (i) $\alpha_2 + \alpha_{2,2} + \alpha_3 = 24 - 2\alpha_2 \geqslant 24 - 8 = 16$, which is impossible by (ii). If $\alpha_0 = 2$ and we assume $\mathscr{B}$ contains two 4-tuples (1, 2, 3, 4) and (1, 5, 6, 7), then at most two of the three pairs

(2, 5), (2, 6) and (2, 7) can appear among the α_2 4-tuples. The same holds for the three pairs (3, 5), (3, 6) and (3, 7) and the three pairs (4, 5), (4, 6) and (4, 7). Consequently, we must have $\alpha_2 \leqslant 6$. But then, from (i), we have $\alpha_2 + \alpha_{2,2} + \alpha_3 = 24 - 2\alpha_2 \geqslant 12$, which is impossible. If $\alpha_0 = 3$, we may assume without loss of generality that $\mathscr{B}$ contains the three 4-tuples (1, 2, 3, 4), (1, 5, 6, 7) and (2, 5, 8, 9). It follows that any of the α_2 4-tuples must contain one of (3, 6), (3, 7), (4, 6) or (4, 7). Thus $\alpha_2 \leqslant 4$ and in this case we obtain from (i) $\alpha_2 + \alpha_{2,2} + \alpha_3 = 18 - 2\alpha_0 \geqslant 18 - 8 = 10$, which is impossible by (ii). The proof of the theorem is complete.

4. Existence Theorems

In this section we shall prove that for every integer $n \geqslant 10$, there are several non-isomorphic extended (2, 4)-designs of order n. The auxiliary designs and other results given in Section 2 will be used in our constructions.

THEOREM 4.1. *If* $n \equiv 1$ *or* 4 (mod 12), $n \geqslant 4$, *then* $\{n; k, 0, 3k\}$ *exists, where* $k = 0$ *or* 1.

Proof. Let (X, B) be a BIBD $B(4, 1; n)$ which exists by [5]. Let $B' = \{(x, x, x, x) \mid x \in X\}$ and put $\mathscr{B} = B \cup B'$. Then it is readily seen that $(X, \mathscr{B})$ is in $\{n; 0, 0, 0\}$ and contains at least one copy of D_4. By removing a copy of D_4 and replacing it by a copy of D_4^*, we obtain the desired result.

THEOREM 4.2. *If* $n \equiv 4$ (mod 12), *then* $\{n; k, 0, 3k\}$ *exists, where* $k = 0, 1, \ldots, n/4$.

Proof. Let (X, B) be a resolvable BIBD $B(4, 1; n)$, which exists by [6]. Let $B' = \{(x, x, x, x) \mid x \in X\}$ and put $\mathscr{B} = B \cup B'$. Then $(X, \mathscr{B}) \in \{n; 0, 0, 0\}$ contains $n/4$ mutually disjoint copies of D_4 arising from the blocks in a parallel class of B. The existence of D_4^* allows us to apply Lemma 2.8 to obtain the desired result.

THEOREM 4.3. *If* $n \equiv 2$ *or* 8 (mod 12), $n \neq 8$, *then* $\{n; 0, k, (n/2) - k\}$ *exists, where* $k = 0, 1, \ldots, n/2$.

Proof. Let (X, B, G) be a $GD(4, 1, 2; n)$ given by Lemma 2.4, where B denotes the collection of blocks and G the collection of groups of the design. Let $B' = \{(a, a, a, a), (b, b, b, a) \mid (a, b) \in G\}$ and put $\mathscr{B} = B \cup B'$. A direct verification shows that $(X, \mathscr{B}) \in \{n; 0, 0, n/2\}$ contains $n/2$ mutually disjoint copies of D_2. By applying Lemma 2.8 with D_2^* the result follows.

THEOREM 4.4. *If* $n \equiv 5$ *or* 11 (mod 12), $n \neq 11, 17$, *then* $\{n; 0, k, [(n+3)/2] - k\}$ *exists, where* $k = 0, 1, \ldots, (n-3)/2$.

Proof. Let (X, B, G) be a $GD(4, 1, \{2, 5^*\}; n)$, which exists by Lemma 2.5. Let $\mathscr{B}$ consist of (i) the blocks of B, (ii) a copy of D_2 on each group of size two in G, (iii) a copy of D_5 on the unique group of size five in G. Then it is readily checked that $(X, \mathscr{B}) \in \{n; 0, 0, (n+3)/2\}$ and, since D_5 contains a copy of D_2, $(X, \mathscr{B})$ contains $(n-5)/2 + 1 = (n-3)/2$ mutually disjoint copies of D_2. By applying Lemma 2.8 with D_2^*, we obtain the required result.

THEOREM 4.5. *If* $n \equiv 2$ *or* 5 (mod 12), *then* $\{n; 0, k, n-1-k\}$ *exists, where* $k = 0$ *or* 1.

Proof. Let (X, B) be a BIBD $B(4, 1; n-1)$. Let $X^* = X \cup \{\infty\}$ where $\infty \notin X$. Let $B' = \{(x, x, x, \infty), (\infty, \infty, \infty, \infty) \mid x \in X\}$. Put $\mathscr{B} = B \cup B'$. Then it is easily seen that $(X^*, \mathscr{B})$ is in $\{n; 0, 0, n-1\}$ and contains at least one copy of D_2. The result follows from Lemma 2.8.

THEOREM 4.6. *If* $n \equiv 0$ *or* 3 (mod 12), *then* $\{n; n/3 - k, 0, 3k\}$ *exists, where* $k = 0, 1, \ldots, n/3$.

Proof. Let (X, B) be a BIBD $B(4, 1; n+1)$ and delete a particular point $\infty \in X$ from the design. We thus obtain a $GD(4, 1, 3; n)(X^*, B', G)$ where $X^* = X - \{\infty\}$, G denotes the collection of truncated blocks of B, and B' the remaining blocks of B. Now $GD(4, 1, 3; n)$ is a special type of PBD $B(\{3, 4\}, 1; n)$ where the blocks of size three are mutually disjoint and thus possess distinct representatives. By applying Lemma 2.7, we may obtain a design $(X^*, \mathscr{B}) \in \{n; n/3, 0, 0\}$ which contains $n/3$ mutually disjoint copies of D_3. By further applying Lemma 2.8 to $(X^*, \mathscr{B})$, the result follows.

THEOREM 4.7. *If* $n \equiv 6$ *or* 9 (mod 12), $n \neq 6, 9, 18$, *then* $\{n; (n+3)/3 - k, 0, 3k\}$ *exists, where* $k = 0, 1, \ldots, (n-6)/3$.

Proof. Let (X, F) be a minimal (2, 4)-covering of order $n+1$ as in Lemma 2.2. Let $X = \{\infty, 1, 2, \ldots, n\}$ and assume the pair $(\infty, 1)$ appears in four of the 4-tuples of F and also assume that the collection of 4-tuples containing ∞ is given by

$$F' = \{(\infty, 1, 2, 3), (\infty, 1, 4, 5), (\infty, 1, 6, 7), (\infty, 1, 8, 9), (\infty, 10, 11, 12),$$
$$(\infty, 13, 14, 15), \ldots, (\infty, n-2, n-1, n)\}.$$

Now if we delete the point ∞ from the covering (X, F), we shall obtain a pairwise balanced design $B(\{3, 4\}, 1; n)$ (X^*, B) where $X^* = \{1, 2, \ldots, n\}$ and B

consists of the blocks of (X, F) not truncated by the removal of ∞ together with the truncated blocks of size three given by

$$B' = \{(1, 2, 3), (1, 4, 5), (1, 6, 7), (1, 8, 9), (10, 11, 12), (13, 14, 15), \ldots, (n-2, n-1, n)\}.$$

It is fairly obvious that the blocks of B' possess distinct representatives—in fact the following is a set of distinct representatives: $S = \{2, 4, 6, 8, 10, 13, \ldots, n-2\}$. By Lemma 2.7, we may obtain from (X^*, B) an extended (2, 4) design $(X^*, \mathscr{B}) \in \{n; (n+3)/3, 0, 0\}$ which contains $(n-6)/3$ mutually disjoint copies of D_3. By further applying Lemma 2.8 to $(X^*, \mathscr{B})$, replacing D_3 by D_3^*, we obtain the desired result.

Remark 4.8. If in the proof of Theorem 4.7 we select $S^* = \{1, 4, 6, 8, 10, 13, \ldots, n-2\}$ as our set of distinct representatives for the blocks B' we would obtain a design $(X^*, \mathscr{B}^*) \in \{n; (n+3)/3, 0, 0\}$ non-isomorphic to that for which the selection is defined by the set $S = \{2, 4, 6, 8, 10, 13, \ldots, n-2\}$.

THEOREM 4.9. *If* $n \equiv 6$ *or* 9 (mod 12), $n \neq 6, 9, 18$, *then* $\{n; n/3 + 7 - k, 0, 3k\}$ *exists, where* $k = 0, 1, \ldots, n/3 - 7$.

Proof. Let (X, B) be a pairwise balanced design $B(\{4, 7^*\}, 1; n+1)$, which exists by Lemma 2.3. Let X' be the subset of X on which the unique block of size seven is based. We shall delete from the PBD (X, B) a particular point $\infty \in X - X'$ to obtain a group divisible design $GD(\{4, 7^*\}, 1, 3; n)$ (X^*, B^*, G) where $X^* = X - \{\infty\}$, G consists of the truncated blocks of B and B^* the remaining blocks of B. In all G contains $n/3$ disjoint blocks of size three. Thus we may select any point in a group as the representative of that group to obtain a set of distinct representatives of G. Now we shall specifically select a point in each group which is not in X' as the representative for that group. Let $\mathscr{B}$ consist of (i) the blocks of size four of B^*, (ii) the blocks of a D_7 based on X', (iii) the blocks (a, a, b, c), (b, b, b, b), (c, c, c, c) where $a, b, c \in X - X'$ and a is the representative of $(a, b, c) \in G$, and (iv) the blocks (x, x, y, z), (y, y, y, y) where $x, y \in X - X'$, $z \in X'$ and x is the representative of $(x, y, z) \in G$. A direct verification shows that $(X^*, \mathscr{B}) \in \{n; n/3 + 7, 0, 0\}$ and that $(X^*, \mathscr{B})$ contains $n/3 - 7$ mutually disjoint copies of D_3. Finally, we apply Lemma 2.8 to complete the proof.

THEOREM 4.10. *If* $n \equiv 2$ *or* 11 (mod 12), *then* $\{n; (2n-4)/3, k, 1-k\}$ *exists, where* $k = 0$ *or* 1.

Proof. Let (X, B) be a BIBD $B(4, 1; n+2)$. We delete two points from (X, B) to obtain a pairwise balanced design $B(\{2, 3, 4\}, 1; n)(X^*, B^*)$ which contains one block of size two disjoint from $(2n-4)/3$ blocks of size three.

The blocks of size three cover $n - 2$ points and each of these points appears exactly twice among the collection of these blocks. Thus we can always find a set of distinct representatives of the blocks of sizes two and three in B^*. By Lemma 2.7 we can obtain an extended (2, 4)-design $(X^*, \mathscr{B}) \in \{n; (2n - 4)/3, 0, 1\}$ which contains a copy of D_2. The result follows by replacing D_2 by D_2^*.

THEOREM 4.11. *If $n \equiv 5$ or 8 (mod 12), $n \neq 8, 17$, then $\{n; (2n - 10)/3, k, 1 - k\}$ exists where $k = 0$ or 1.*

Proof. Let (X, B) be a pairwise balanced design $B(\{4, 7^*\}, 1; n + 2)$. We delete two points from (X, B) which lie in the unique block of size seven to obtain a pairwise balanced design $B(\{3, 4, 5^*\}, 1; n)$ (X^*, B^*) which contains $(2n - 10)/3$ blocks of size three disjoint from the unique block of size five. The blocks of size three cover $n - 5$ points of X^* and each of these points appears exactly twice among the size three blocks. We can readily find a set S of distinct representatives for the size three blocks. Let $\mathscr{B}$ consist of (i) the blocks of size four in B^*, (ii) the blocks (a, a, b, c) where $a \in S$ is the representative of $(a, b, c) \in B^*$, (iii) the blocks of a D_5 based on the points F of the unique block of size five, and (iv) the blocks (x, x, x, x) where $x \notin S \cup F$. A direct verification shows that $(X^*, \mathscr{B}) \in \{n; (2n - 10)/3, 0, 4\}$ contains a copy of D_5 and the result follows from Lemma 2.8.

THEOREM 4.12. *If $n \equiv 5$ or 8 (mod 12), $n \neq 5, 8, 17$, then $\{n; (2n - 16)/3, k, 4 - k\}$ exists, where $k = 0, 1, 2, 3, 4$.*

Proof. Let (X, F) be a minimal (2, 4)-covering of order $n + 2$ as in Lemma 2.2. Let $X = \{\infty_1, \infty_2, 1, 2, \ldots, n\}$ and assume without loss of generality that F contains the four 4-tuples $(\infty_1, \infty_2, 1, 2)$, $(\infty_1, \infty_2, 3, 4)$, $(\infty_1, \infty_2, 5, 6)$, $(\infty_1, \infty_2, 7, 8)$. This means that all pairs distinct from (∞_1, ∞_2) appear exactly once among the 4-tuples of F. Now we delete from (X, F) the two points ∞_1, ∞_2 to form a pairwise balanced design $B(\{2, 3, 4\}, 1; n)(X^*, B^*)$ where $X^* = \{1, 2, \ldots, n\}$ and B^* contains the four disjoint blocks of size two $(1, 2)$, $(3, 4)$, $(5, 6)$, $(7, 8)$, each disjoint from $(2n - 16)/3$ blocks of size three. The blocks of size three cover $n - 8$ points and each of these points appears exactly twice among the size three blocks. Clearly, any collection of t blocks of size three and s blocks of size two in B^* must contain at least $(3t + 2s)/2 \geqslant t + s$ points. In other words, the blocks of sizes two and three possess distinct representatives. By applying Lemma 2.7 to (X^*, B^*), we may obtain an extended (2, 4)-design $(X^*, \mathscr{B}) \in \{n; (2n - 16)/3, 0, 4\}$ such that $(X^*, \mathscr{B})$ contains 4 mutually disjoint copies of D_2. The result follows from Lemma 2.8.

THEOREM 4.13. *If $n \equiv 1$ or 10 (mod 12), then $\{n; n - 1, 0, 0\}$ exists.*

Proof. The statement is obviously true for $n = 1$. In what follows assume $n \geqslant 10$. Let (X, B) be a BIBD $B(4, 1; n + 3)$. We delete three points of (X, B) which are all contained in the same block in order to obtain a PBD $B(\{3, 4\}, 1; n)(X^*, B^*)$. In (X^*, B^*) there are $n - 1$ blocks of size three covering $n - 1$ points and each of these points appear exactly three times among the size three blocks. Any collection of t of these blocks must contain at least $3t/3 = t$ points. And so the size three blocks possess distinct representatives. Applying Lemma 2.7, the desired result is obtained.

THEOREM 4.14. *If* $n \equiv 4$ *or* 7 (mod 12), $n \neq 7, 16$, *then* $\{n; n - 4 + k, 0, 3k\}$ *exists where* $k = 0$ *or* 1.

Proof. If $n = 4$, then D_4 and D_4^* provide a proof. Let us then assume $n \geqslant 19$ in what follows. Let (X, B) be a PBD $B(\{4, 7^*\}, 1; n + 3)$. We delete three points from (X, B) which are contained in the unique block of size seven to obtain a PBD $B(\{3, 4\}, 1; n)(X^*, B^*)$ in which there are $n - 4$ blocks of size three. The size three blocks cover $n - 4$ points and each of these points appears exactly three times among the blocks. Now any collection of t of these blocks must contain at least $3t/3 = t$ points. Thus we can find distinct representatives for the blocks of size three and obtain an extended (2, 4)-design $(X^*, \mathscr{B}) \in \{n; n - 4, 0, 0\}$. $(X^*, \mathscr{B})$ contains a copy of D_4 based on the block of size four remaining after truncating the unique block of size seven and the result follows by applying Lemma 2.9 to $(X^*, \mathscr{B})$.

THEOREM 4.15. *If* $n \equiv 1$ *or* 10 (mod 12), $n \geqslant 10$, *then* $\{n; n - 4, 0, 3\}$ *exists.*

Proof. Let (X, B) be a BIBD $B(4, 1; n + 3)$. We select three points which do not all lie in the same block of B and delete these points from (X, B) to obtain a PBD $B(\{2, 3, 4\}, 1; n)(X^*, B^*)$ which contains 3 mutually disjoint blocks of size two and $n - 4$ blocks of size three. Each of the 6 points covered by the blocks of size two appears exactly once among the blocks of size three, while each of the remaining $n - 7$ points covered by the size three blocks appear exactly three times among these blocks. We claim that any collection of t blocks of size three and s blocks of size two must contain $t + s$ points, where $0 \leqslant t \leqslant n - 4$ and $0 \leqslant s \leqslant 3$. First of all, if $s = 0$ or $t = 0$, the claim is easily justified. In what follows we assume $1 \leqslant t \leqslant n - 4$ and $1 \leqslant s \leqslant 3$. Let p be the number of points covered by the $t + s$ blocks. Since $1 \leqslant s \leqslant 3$, there are at least two points (from a size two block) which are covered at most twice by the $t + s$ blocks. Consequently, $p > (3t + 2s)/3 = t + s - s/3$. Since p is an integer and $1 \leqslant s \leqslant 3$, we must have $p \geqslant t + s$ as claimed. It follows that the size two and size three blocks possess distinct representatives and the result follows from Lemma 2.7.

THEOREM 4.16. *If* $n \equiv 4$ *or* 7 (mod 12), $n \geqslant 19$, *then* $\{n; n-5, 0, 3\}$ *exists.*

Proof. Let (X, F) be a minimal (2, 4)-covering of order $n + 3$ as in Lemma 2.2. Let $X = \{\infty_1, \infty_2, \infty_3, 1, 2,..., n\}$ and assume without loss of generality that F contains the four 4-tuples $(\infty_1, \infty_2, \infty_3, 1)$, $(\infty_1, \infty_2, 2, 3)$, $(\infty_1, \infty_2, 4, 5)$, $(\infty_1, \infty_2, 6, 7)$. We delete the three points ∞_1, ∞_2, ∞_3 from (X, F) in order to obtain PBD $B(\{2, 3, 4\}, 1; n)(X^*, B^*)$ which contains the three mutually disjoint blocks (2, 3), (4, 5), (6, 7) and $n - 5$ blocks of size three. The situation is similar to that in the proof of Theorem 4.15. It is readily checked that the blocks of sizes two and three possess distinct representatives. We thus obtain the desired result by applying Lemma 2.7.

Finally, for the case $n = 18$ we have the following result.

THEOREM 4.17. *There exist at least three inequivalent extended* (2, 4)-*designs of order* 18.

Proof. Let $X = \{1, 2,..., 18\}$ and let $\mathscr{B} = \{(1, 1, 1, 5), (2, 2, 2, 5), (3, 3, 3, 5), (4, 4, 4, 5), (5, 5, 5, 5), (6, 6, 6, 10), (7, 7, 7, 10), (8, 8, 8, 10), (9, 9, 9, 10), (10, 10, 10, 5), (11, 11, 3, 7), (12, 12, 1, 9), (13, 13, 4, 6),$

TABLE I

n(mod 12)	$\{n; \alpha_2, \alpha_{2,2}, \alpha_3\}$
0	$\{n; n/3 - k, 0, 3k\}$, $k = 0, 1,..., n/3$.
1	$\{n; n-4, 0, 3\}$, $n \geqslant 13$; $\{n; n-1, 0, 0\}$ and $\{n; 0, 3k\}$, $k = 0, 1$, $n \neq 1$.
2	$\{n; (2n-4)/3, k, 1-k\}$, $k = 0, 1$; $\{n; 0, j, n-1-j\}$, $j = 0$ or 1 and $\{n; 0, s, n/2 - s\}$, $s = 0, 1,..., n/2$.
3	$\{n; n/3, 0, 3k\}$, $k = 0, 1,..., n/3$.
4	$\{n; n-5, 0, 3\}$, $n \geqslant 28$; $\{n; n-4+k, 0, 3k\}$, $k = 0$ or 1, $n \neq 16$; $\{n; s, 0, 3s\}$, $s = 0, 1,..., n/4$.
5	$\{n; (2n-16/3, k, 4-k\}$, $k = 0, 1, 2, 3, 4$, $n \neq 5, 17$; $\{n; (2n-10)/3, j, 1-j\}$, $j = 0$ or 1, $n \neq 17$; $\{n; 0, s, n-1-s\}$, $s = 0$ or 1 and $\{n; 0, t, (n+3)/2 - t\}$, $t = 0, 1,..., (n-3)/2$, $n \neq 17$.
6	$\{18; 8, 1, 8\}$, $\{18; 8, 0, 9\}$; $\{n; n/3 + 7 - k, 0, 3k\}$, $k = 0, 1,..., n/3 - 7$, $n \neq 6, 18$; $\{n; (n+3)/3 - j, 0, 3j\}$, $j = 0, 1,..., (n-6)/3$, $n \neq 6, 18$.
7	$\{n; n-5, 0, 3\}$, $n \geqslant 19$; $\{n; n-4+j, 0, 3j\}$, $j = 0$ or 1, $n \neq 7$.
8	$\{n; (2n-16)/3, k, 4-k\}$, $k = 0, 1, 2, 3, 4$; $n \neq 8$; $\{n; (2n-10/3, j, 1-j\}$, $j = 0$ or 1, $n \neq 8$; $\{n; 0, s, n/2 - s\}$, $s = 0, 1,..., n/2$, $n \neq 8$.
9	$\{n; n/3 + 7 - k, 0, 3k\}$, $k = 0, 1,..., n/3 - 7$, $n \neq 9$; $\{n; (n+3)/3 - j, 0, 3j\}$, $j = 0, 1,..., (n-6)/3$, $n \neq 9$.
10	$\{n; n-4, 0, 3\}$, and $\{n; n-1, 0, 0\}$.
11	$\{n; (2n-4)/3, k, 1-k\}$, $k = 0$ or 1; $\{n; 0, j, (n+3)/2 - j\}$, $j = 0, 1,..., (n-3)/2$, $n \neq 11$.

(14, 14, 2, 8), (15, 15, 2, 6), (16, 16, 3, 9), (17, 17, 1, 8), (18, 18, 4, 7), (1, 2, 3, 4), (1, 6, 11, 16), (1, 7, 13, 15), (1, 10, 14, 18), (2, 7, 12, 17), (2, 9, 11, 18), (2, 10, 13, 16), (3, 6, 14, 17), (3, 8, 13, 18), (3, 10, 12, 15), (4, 8, 12, 16), (4, 9, 14, 15), (4, 10, 11, 17), (5, 6, 12, 18), (5, 7, 14, 16), (5, 8, 11, 15), (5, 9, 13, 17), (6, 7, 8, 9), (11, 12, 13, 14), (15, 16, 17, 18)}. Then it is readily checked that $(X, \mathscr{B}) \in \{18; 8, 0, 9\}$ and contains a copy of D_5 based on the set $\{1, 2, 3, 4, 5\}$. Let $\mathscr{B}'$ and $\mathscr{B}^*$ be defined by $\mathscr{B}' = [\mathscr{B} - \{(5, 5, 5, 5), (10, 10, 10, 5)\}] \cup \{(5, 5, 10, 10)\}$ and $\mathscr{B}^* = [\mathscr{B} - \{(4, 4, 4, 5), (5, 5, 5, 5)\}] \cup \{(4, 4, 5, 5)\}$. Then $(X, \mathscr{B}') \in \{18; 8, 1, 8\}$ and contains no copy of D_5 or D_5^*, while $(X, \mathscr{B}^*) \in \{18; 8, 1, 8\}$ and contains a copy of D_5^*. The designs $(X, \mathscr{B})$, $(X, \mathscr{B}')$ and $(X, \mathscr{B}^*)$ are clearly inequivalent.

We summarize the results of this section in Table I.

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